Name: _____   RED ID: _____

Fall 2023 Math 245 Final Exam

Please read the following directions:

Please write legibly, with plenty of white space. Please **print** your name and REDID in the designated spaces above. Please fit your answers into the designated areas; material outside the designated areas (such as on this cover page) will not be graded. To get credit, you must also show adequate work to justify your answers. If unsure, show the work. The first four questions are worth 6-12 points, and the remaining sixteen questions are worth either 9-18 or 10-20 points, as indicated. The maximum possible score is $4 \times 12 + 9 \times 18 + 7 \times 20$, for a total of 350 points. The use of notes, books, calculators, or other materials on this exam is strictly prohibited, except you may bring one 8.5"x11" page (both sides) with your handwritten notes. If you need scratch paper, you may use any blank space on your note sheet and on this front page. This exam will begin at 10:30 and will end at 12:30; pace yourself accordingly. Good luck!

Special exam instructions for HH-130:
1. Please stow all bags/backpacks/boards at the front of the room. All contraband, except phones, must be stowed in your bag. All smartwatches and phones must be silent, non-vibrating, and either in your pocket or stowed in your bag.
2. Please remain quiet to ensure a good test environment for others.
3. Please keep your exam on your desk; do not lift it up for a better look.
4. If you have a question or need to use the restroom, please come to the front. Bring your exam. I cannot come to you unless you are sitting by an aisle.
5. If you are done and want to submit your exam and leave, please wait until one of the designated exit times, listed below. Please do **NOT** leave at any other time. If you are sure you are done, just sit and wait until the next exit time, with this cover sheet visible.

Designated exam exit times:
   10:50 "See you next semester"
   11:10 "I wish I had studied more"
   11:30 "One extra hour of drinking – worth it"
   11:50 "Maybe this will be good enough"
   12:10 "There is nothing more in my brain, let me out of here"
   12:30 "I need every second I can get"

Problem 1. Carefully state the following definitions:
  a. power set

  b. trichotomous

Problem 2. Carefully state the following definitions:
  a. $\equiv_n$

  b. S/R [HINT: S is a set.]

Problem 3. Carefully state the following definitions:
  a. lower bound

  b. product order

Problem 4. Carefully state the following definitions:
  a. function

  b. injection

(9-18pts) Problem 5. Prove or disprove: $\exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, \ y < x$.

(9-18pts) Problem 6. Let $x \in \mathbb{R}$. Prove that if $x^2 + 1$ is irrational, then $2x$ is irrational.

(10-20pts) Problem 7. Let p, q be propositions. Use a truth table to prove that $p \uparrow (p \uparrow q) \equiv p \to (p \wedge q)$.

(10-20pts) Problem 8. Prove that for every $n \in \mathbb{N}$, we must have $\displaystyle\sum_{i=-1}^{n} (2i + 1) = n^2 + 2n$.

(9-18pts) Problem 9. Let $a_n = 0.5 + \sin n$. Prove or disprove that $a_n = O(1)$.

(9-18pts) Problem 10. Find, with justification, all integers $x \in [0, 100)$ satisfying $20x \equiv 50 \pmod{100}$.

(10-20pts) Problem 11. Prove or disprove: For all sets A, B, C, if $A \cup B \subseteq B \cap C$, then $A \subseteq C$.

(10-20pts) Problem 12. Find, with justification, a relation from $\{a, b, c\}$ to $\{d, e, f\}$ that is right-total, left-total, and not right-definite.

For problems 13-16, we let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, and consider the relation on S given by $R = \{(x, y) : (x = y) \vee (x^2 | y)\}$

(9-18pts) Problem 13. Determine, with justification, whether R is reflexive, irreflexive, and transitive.

(9-18pts) Problem 14. Prove that R is a partial order. [No need to repeat any proof from #13.]

(10-20pts) Problem 15. Draw the Hasse diagram for this poset. Determine how many elements are maximum, maximal, minimum, and minimal.

(9-18pts) Problem 16. Determine, with justification, the width and height of this poset.

(10-20pts) Problem 17. Define relation R on $\mathbb{Q}$ via $R = \{(a,b) : a^2 - b^2 \in \mathbb{Z}\}$. Prove that R is an equivalence relation.

(10-20pts) Problem 18. Set $S = \{x \in \mathbb{R} : x > 0\}$. Consider the function $f : \mathbb{R} \to S$ given by $f : x \mapsto e^{-(x/2)}$. Prove that f is a bijection.

(9-18pts) Problem 19. Let R be an equivalence relation on S. Suppose $x, y, z \in S$ with $z \in [x] \cap [y]$. Prove that $[x] \subseteq [y]$. This is (half of) a book theorem. Do not use the theorem to prove itself.

(9-18pts) Problem 20. Choose S to be $\mathbb{R}$, or $\mathbb{Z}$, or $\{x \in \mathbb{R} : x > 0\}$. Then find, with justification, any non-identity function $f : S \to S$ satisfying $f(x) = f^{-1}(x)$. [Not $f(x) = x$; that's forbidden.]